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Embedded monopoles

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Abstract

Using the embedded defect method, we classify the possible embeddings of a 't Hooft–Polyakov monopole in a general gauge theory. We then discuss some similarities with embedded vortices and relate our results to fundamental monopoles.

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1. Introduction

When describing classical solutions to spontaneously broken Yang–Mills theories there are several available methods. If the solutions are topologically stable there is a topological classification [1], while underneath this there are finer classifications relating to the degeneracy of classical solutions. These are important for properties other than stability—for instance Goddard, Nuyts and Olive use such a classification of non-Abelian monopoles to conjecture a dual gauge group [2].

For non-Abelian monopoles a finer than topological classification can be achieved by using fundamental monopoles [3]. These allow any monopole to be described as a composite of several fundamental monopoles, each of which is associated with a simple root of the gauge group. Furthermore, in the BPS limit each fundamental monopole is known to take an Prasad–Sommerfield form on the $su(2)$ algebra defined by its root [3].

For defects in general (BPS and non-BPS) a good formalism to describe a finer than topological classification is the embedded defect method of Barriola, Vachaspati, and Bucher [4]. This describes how a solution in one gauge theory is embedded in a larger theory and gives some constraints for when this is allowed. Of particular interest are embedded vortices and embedded monopoles.

For embedded vortices the embedding conditions of Ref. [4] can be expressed in a very simple way that directly relates to the geometry of the vacuum manifold [5]. It can then be shown that embedded vortices separate into classes under the action of the residual gauge symmetry and from this easily specify their gauge degeneracy.

Such a picture has not been applied to embedded monopoles. For that reason this Letter aims to:

(a) Apply embedded defect methods to monopoles.

(b) Find similarities between embedded monopoles and embedded vortices.

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(c) Describe how these results on embedded monopoles relate to fundamental monopoles.

In addition, this gives some new results and a new picture of non-Abelian monopoles, both of which may be useful when examining their properties.

2. Formalism

Consider a spontaneously broken Yang–Mills theory with a compact semi-simple gauge group, G, and a scalar field, $\Phi \in \mathcal{V}$, in the D representation of G

$$\mathcal{L}[\Phi, A^\mu] = -\tfrac{1}{4}\langle F_{\mu\nu}, F^{\mu\nu}\rangle + \tfrac{1}{2}\langle D_\mu \Phi, D^\mu \Phi\rangle - V[\Phi]. \tag{1}$$

Here the covariant derivative is

$$D^\mu \Phi = \partial^\mu \Phi + d(A^\mu)\Phi,$$

while the field tensor is

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + [A^\mu, A^\nu],$$

a covariant curl. The derived representation d is defined by $e^{d(X)} = D(e^X)$, where X is in the Lie algebra $\mathcal{G}$.

In this Letter we use a coordinate independent notation, which we believe better reflects the geometry behind most of our results [5,6]. Then the gauge kinetic term in (1) is defined by an inner product on $\mathcal{G}$

$$\langle X, Y\rangle = -\frac{2}{f_a^2}\operatorname{tr}[\operatorname{ad}(X)\operatorname{ad}(Y)], \quad X, Y \in \mathcal{G}_a, \tag{2}$$

with f_a a coupling constant for each simple subgroup $G_a \subseteq G$. Likewise, the scalar kinetic term is defined by a Euclidean inner product $\langle \Phi_1, \Phi_2\rangle$ on $\mathcal{V}$.[1]

In this notation the field equations are

$$D_\mu D^\mu \Phi = -\frac{\partial V}{\partial \Phi}, \qquad D_\mu F^{\mu\nu} = J^\nu, \tag{3}$$

with current

$$\langle J^\nu, Y\rangle = \langle d(Y)\Phi, D^\nu \Phi\rangle - \langle D^\nu \Phi, d(Y)\Phi\rangle$$

and covariant derivative

$$D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}].$$

[1] We have used same notation for both inner products, as the one referred to should be clear from the context.

If the potential $V[\Phi]$ is minimized at some value Φ_0 the residual gauge symmetry is

$$H = \{h \in G : D(h)\Phi_0 = \Phi_0\}, \tag{4}$$

with $\mathcal{H}$ the Lie algebra of H. This defines an $\operatorname{Ad}(H)$-invariant decomposition of $\mathcal{G}$ into massless and massive gauge boson generators

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M}, \qquad \operatorname{Ad}(H)\mathcal{H} \subseteq \mathcal{H}, \qquad \operatorname{Ad}(H)\mathcal{M} \subseteq \mathcal{M}. \tag{5}$$

Here Ad refers to the adjoint representation of G on $\mathcal{G}$, with $\operatorname{Ad}(g)X = gXg^{-1}$ and derived representation $\operatorname{ad}(X)Y = [X, Y]$ for $X, Y \in \mathcal{G}$.

A central feature of this and some related papers [5,6] is the reduction of $\mathcal{M}$ into irreducible subspaces under the adjoint action of H. These correspond to irreducible representations of H on $\mathcal{M}$,

$$\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n. \tag{6}$$

Physically, each $\mathcal{M}_a$ defines a gauge multiplet of massive gauge bosons (for instance the W- and Z-bosons in electroweak theory).

Embedded defects [4] are $(2-k)$-dimensional topological defects that remain a solution when they are embedded into a larger gauge theory (with $k = 0, 1, 2$ for domain walls, vortices and monopoles). They are defined by an inclusion of their gauge symmetry breaking in that of the full theory

$$\begin{array}{ccc} G & \to & H \\ \cup & & \cup \\ G_{\text{emb}} & \to & H_{\text{emb}} \end{array} \qquad \pi_k(G_{\text{emb}}/H_{\text{emb}}) \neq 0. \tag{7}$$

In the original Ref. [4] these embedded defects have fields fully embedded over the spatial domain

$$\Phi_{\text{emb}}(x) \in \mathcal{V}_{\text{emb}}, \qquad A^\mu_{\text{emb}}(x) \in \mathcal{G}_{\text{emb}}, \qquad x \in \mathbb{R}^{1+k}, \tag{8}$$

where $\mathcal{V}_{\text{emb}}$ is a vector subspace of $\mathcal{V}$ and

$$D(G_{\text{emb}})\mathcal{V}_{\text{emb}} = \mathcal{V}_{\text{emb}}. \tag{9}$$

Here we find the constraint (9) too restrictive (see discussion around (35) below). Therefore, we also consider a more general set of asymptotically embedded defects that as $x \to \infty$ have

$$\Phi \sim \Phi_{\text{emb}}(x) \in \mathcal{V}_{\text{emb}}, \qquad A^\mu \sim A^\mu_{\text{emb}}(x) \in \mathcal{G}_{\text{emb}}. \tag{10}$$

These asymptotically coincide with an embedded defect, but may differ from that form (8) elsewhere.

An embedded defect is a solution of the full theory if the field equations reduce to consistent field equations on the defect's embedded theory [4]. This gives four constraints from the two field equations in (3):

(a) The current, evaluated from $\Phi_{\rm emb}$ and $A^{\mu}_{\rm emb}$, satisfies

$$\langle J^{\nu}, \mathcal{G}^{\perp}_{\rm emb}\rangle = 0, \tag{11}$$

with $\mathcal{G} = \mathcal{G}_{\rm emb} \oplus \mathcal{G}^{\perp}_{\rm emb}$. This constrains $\mathcal{G}_{\rm emb} \subseteq \mathcal{G}$.

(b) The kinetic scalar term satisfies

$$\langle D_{\mu} D^{\mu} \Phi_{\rm emb}, \mathcal{V}^{\perp}_{\rm emb}\rangle = 0, \tag{12}$$

with $\mathcal{V} = \mathcal{V}_{\rm emb} \oplus \mathcal{V}^{\perp}_{\rm emb}$. While this is the case when (9) is satisfied, it is otherwise very restrictive and generally only holds asymptotically for certain parameter values.

(c) The scalar potential, evaluated for $\Phi_{\rm emb}$, satisfies

$$\left\langle \frac{\partial V}{\partial \Phi}, \mathcal{V}^{\perp}_{\rm emb}\right\rangle = 0. \tag{13}$$

This constrains the potential—for instance it holds in the BPS limit.

(d) The gauge kinetic terms satisfies

$$\langle D_{\mu} F^{\mu\nu}, \mathcal{G}^{\perp}_{\rm emb}\rangle = 0. \tag{14}$$

Eq. (14) always holds by algebraic closure of $\mathcal{G}_{\rm emb}$, as observed in [4].

3. Embedded monopoles

By (7) an embedded monopole is defined by embedding an $su(2) \to u(1)$ symmetry breaking in that of the full theory

$$\begin{array}{ccc} \mathcal{G} & \to & \mathcal{H} \\ \cup & & \cup \\ su(2) & \to & u(1) \end{array} \tag{15}$$

Then the embedded monopole has a 't Hooft–Polyakov form [7] on the $su(2)$ subtheory

$$\Phi(\boldsymbol{r}) = \frac{H}{r} D\big(g(\hat{\boldsymbol{r}})\big)\Phi_0,$$
$$A^i(\boldsymbol{r}) = \frac{K-1}{r}\epsilon_{iab}\hat{r}_b t_a, \tag{16}$$

with $su(2)$ basis $[t_a, t_b] = \epsilon_{abc} t_c$ and, in spherical polars, $g(\hat{\boldsymbol{r}}) = e^{\varphi t_3} e^{\vartheta t_2} e^{-\varphi t_3}$. Likewise, an asymptotically embedded monopole only has its asymptotic fields similar to (16), where [8]

$$H - r = \mathrm{O}\big[\exp(-\mu r)\big],$$
$$K = \mathrm{O}\big[\exp(-m r)\big]. \tag{17}$$

Here μ and m are the scalar and gauge boson masses in the embedded $su(2)$ subtheory. Elsewhere the fields of an asymptotically embedded monopole may differ from the embedded monopole.

Both an embedded and asymptotically embedded monopole has a long range magnetic field[2]

$$-\frac{1}{2}\epsilon_{ijk} F^{jk} \sim \frac{\hat{r}_i}{r^2} M(\hat{\boldsymbol{r}}), \quad M(\hat{\boldsymbol{r}}) = \mathrm{Ad}\big(g(\hat{\boldsymbol{r}})\big)t_3. \tag{18}$$

The scalar field asymptotically tends to the vacuum with $\Phi \sim \Phi_0$ in the $\hat{x}_3$-direction. Also in that direction the magnetic generator $M = M(\hat{z})$ is an element of $\mathcal{H}$, and satisfies the topological quantization $\exp(4\pi M) = 1$ [9].

To simplify the following calculations we express (16) in a unitary gauge (so $\mathcal{V}_{\rm emb} = \mathbb{R}\Phi_0$). This is achieved with a gauge transformation

$$\Phi \mapsto D\big(g^{-1}\big)\Phi,$$
$$\boldsymbol{A} \mapsto \mathrm{Ad}\big(g^{-1}\big)\boldsymbol{A} - \big(\boldsymbol{\nabla} g^{-1}\big)g, \tag{19}$$

which takes the embedded monopole (16) to [10][3]

$$\Phi(\boldsymbol{r}) = \frac{H}{r}\Phi_0, \qquad \boldsymbol{A}(\boldsymbol{r}) = -\boldsymbol{A}_D t_3 - \frac{K}{r}\hat{\boldsymbol{\eta}}_s t_s. \tag{20}$$

Here $\boldsymbol{A}_D = \hat{\boldsymbol{\varphi}}(1-\cos\vartheta)/r\sin\vartheta$ is the Dirac gauge potential ($\boldsymbol{\nabla}\cdot\boldsymbol{A}_D = 0$, $\boldsymbol{\nabla}\wedge\boldsymbol{A}_D = \hat{\boldsymbol{r}}/r^2$) and

$$\hat{\boldsymbol{\eta}}_1 = \sin\varphi\hat{\boldsymbol{\vartheta}} + \cos\varphi\hat{\boldsymbol{\varphi}},$$
$$\hat{\boldsymbol{\eta}}_2 = -\cos\varphi\hat{\boldsymbol{\vartheta}} + \sin\varphi\hat{\boldsymbol{\varphi}} \tag{21}$$

are two orthonormal unit vectors orthogonal to $\hat{\boldsymbol{r}}$. To prove (20) we write the gauge field as

$$\boldsymbol{A} = \frac{K-1}{r}\big((\hat{\boldsymbol{\varphi}}\cdot\boldsymbol{t})\hat{\boldsymbol{\vartheta}} - (\hat{\boldsymbol{\vartheta}}\cdot\boldsymbol{t})\hat{\boldsymbol{\varphi}}\big), \tag{22}$$

[2] Note the physical magnetic field is scaled by the inverse couplings f_a through (2).

[3] That $\boldsymbol{A} \sim -\boldsymbol{A}_D$ is from the conventional definition of $B^k_a = -\frac{1}{2}\epsilon_{ijk}F^{ij}_a$, applying this gives $\boldsymbol{B}_3 \sim \boldsymbol{\nabla}\wedge\boldsymbol{A}_D$.

then use the following identities

$$\mathrm{Ad}(g^{-1})\hat{\boldsymbol{\vartheta}} \cdot \boldsymbol{t} = \cos\varphi t_1 + \sin\varphi t_2, \tag{23}$$

$$\mathrm{Ad}(g^{-1})\hat{\boldsymbol{\varphi}} \cdot \boldsymbol{t} = -\sin\varphi t_1 + \cos\varphi t_2, \tag{24}$$

and evaluate

$$\nabla g^{-1} g = \boldsymbol{A}_D t_3 - \frac{1}{r}\mathrm{Ad}(g^{-1}) \times \left((\hat{\boldsymbol{\varphi}} \cdot \boldsymbol{t})\hat{\boldsymbol{\vartheta}} - (\hat{\boldsymbol{\vartheta}} \cdot \boldsymbol{t})\hat{\boldsymbol{\varphi}}\right). \tag{25}$$

Now the task is to find when such a monopole solution is fully or asymptotically embedded. This is determined by the constraints (11) and (12).

To examine the first constraint $\langle J_\nu, \mathcal{G}_{\mathrm{emb}}^{\perp}\rangle = 0$ we start by rewriting (15) as

$$\begin{array}{ccccc} \mathcal{G} & = & \mathcal{H} & \oplus & \mathcal{M} \\ \cup & & \cup & & \cup \\ su(2) & = & u(1) & \oplus & \mathcal{N} \end{array} \qquad \mathcal{N} = \mathbb{R}t_1 \oplus \mathbb{R}t_2. \tag{26}$$

Substituting J^ν below (3) into (11) gives

$$\langle d(\mathcal{G}_{\mathrm{emb}}^{\perp})\Phi_0, D^\nu\Phi\rangle = \langle D^\nu\Phi, d(\mathcal{G}_{\mathrm{emb}}^{\perp})\Phi\rangle = 0. \tag{27}$$

Now $D^0\Phi = 0$, while spatially in a unitary gauge

$$D^i\Phi = (H/r)'\hat{x}^i\Phi_0 + (HK/r^2)d(\hat{\eta}_s^i t_s)\Phi_0. \tag{28}$$

Using this and noting $\langle d(\mathcal{G})\Phi_0, \Phi_0\rangle = 0$ shows (11) is an algebraic constraint upon the embedding (15)

$$\langle d(\mathcal{G}_{\mathrm{emb}}^{\perp})\Phi_0, d(\mathcal{N})\Phi_0\rangle = \langle d(\mathcal{N})\Phi_0, d(\mathcal{G}_{\mathrm{emb}}^{\perp})\Phi_0\rangle = 0. \tag{29}$$

To this we apply a result proved in [5]:

$$\langle d(X_a)\Phi_0, d(Y_b)\Phi_0\rangle = \lambda_a\lambda_b\langle X_a, Y_b\rangle, \qquad \lambda_a = \frac{\|d(X_a)\Phi_0\|}{\|X_a\|}, \quad X_a \in \mathcal{M}_a, \quad Y_b \in \mathcal{M}_b. \tag{30}$$

Therefore, if $\lambda_a \neq \lambda_b$ the monopole embedding in (15) is given by

$$\mathcal{N} \subseteq \mathcal{M}_a, \tag{31}$$

with $\mathcal{M}_a$ a gauge family in (6). If $\lambda_a = \lambda_b$ the embedding can also be between gauge families (see Ref. [5]).

For the second constraint $\langle D_\mu D^\mu \Phi_{\mathrm{emb}}, \mathcal{V}_{\mathrm{emb}}^{\perp}\rangle = 0$ we evaluate the scalar kinetic term in a unitary gauge. As observed above $D^0\Phi_{\mathrm{emb}} = D^0 D^0\Phi_{\mathrm{emb}} = 0$; then we only need to consider the spatial components

$$D^i D^i \Phi_{\mathrm{emb}} = \left[(H/r)'' + 2(H/r)'/r\right]\Phi_0 + \left(HK^2/r^3\right) d(\hat{\eta}_s^i t_s)\, d(\hat{\eta}_s^i t_s)\,\Phi_0. \tag{32}$$

In evaluating this we used the identity

$$\partial^i \hat{\eta}_s^i = \epsilon_{st}\frac{\cos\vartheta - 1}{r\sin\vartheta}\hat{\varphi}^i\eta_t^i \tag{33}$$

and took, from $d([X, Y]) = [d(X), d(Y)]$,

$$d(t_3)\, d(t_1)\,\Phi_0 = d(t_2)\,\Phi_0, \qquad d(t_3)\, d(t_2)\,\Phi_0 = -d(t_1)\,\Phi_0. \tag{34}$$

Therefore, by (32), condition (12) is satisfied when

$$d(t_1)\, d(t_1)\,\Phi_0 + d(t_2)\, d(t_2)\,\Phi_0 \propto \Phi_0. \tag{35}$$

Note this is a very constraining assumption.

Therefore, if both conditions (31) and (35) hold there are embedded monopole solutions like (16). Towards the core their scalar fields vanish and these monopoles are similar to a 't Hooft–Polyakov monopole.

A more usual situation is when (35) does not hold. Before we discussed asymptotically embedded monopoles, which coincide with an embedded monopole only at infinity. From these the question arises whether condition (12) could be satisfied only in the asymptotic region.

To answer this question we consider the asymptotic form of (32). Substituting Eq. (17) we find

$$(H/r)'' + 2(H/r)'/r = \mathrm{O}\left[\exp(-\mu r)/r\right], \tag{36}$$

$$HK^2/r^3 = \mathrm{O}\left[\exp(-2mr)/r^2\right]. \tag{37}$$

Then condition (12) is satisfied if the second term is negligible compared to the first. Exponentials beat powers, so this occurs when the scalar mass μ is less than twice the gauge mass m. When that occurs the asymptotically embedded monopoles are classified only by (31). From now on we examine the first constraint (31).

4. Solution sets of gauge equivalent monopoles

Because each monopole is defined from an $su(2)$ embedding, and these embeddings generally form degenerate sets, we expect the monopoles to have some degeneracy. Our tactic for determining this degeneracy is to consider the action of the residual gauge symmetry H upon the $su(2)$ embedding.

To start we consider a rigid H transformation (no space–time dependence) of a monopole's asymptotics

$$\Phi(\boldsymbol{r}) \mapsto D(h)\Phi(\boldsymbol{r}), \qquad \boldsymbol{A}(\boldsymbol{r}) \mapsto \mathrm{Ad}(h)\boldsymbol{A}(\boldsymbol{r}). \tag{38}$$

By (16) this simply takes $\boldsymbol{t} \mapsto \mathrm{Ad}(h)\boldsymbol{t}$, from which the $su(2)$ embedding moves to

$$su(2) \mapsto \mathrm{Ad}(h)su(2). \tag{39}$$

Therefore, each monopole is an element in a set of gauge-equivalent monopoles. This set is represented by the H-equivalent $su(2)$ embeddings

$$\mathcal{O} \cong \frac{H}{C_H(su(2))}, \tag{40}$$

where $C_H(su(2)) \subset H$ is the centralizer of $su(2)$ in H (which acts trivially on every element in $su(2)$).

We can express this set more transparently by noting that the denominator in (40) satisfies

$$\begin{aligned} &C_H\big(su(2)\big) = C_H(\mathcal{N}), \\ &su(2) = u(1) \oplus \mathcal{N}, \end{aligned} \tag{41}$$

with $C_H(\mathcal{N}) \subset H$ the centralizer of $\mathcal{N}$ in H (which acts trivially on every element in $\mathcal{N} = \mathbb{R}t_1 \oplus \mathbb{R}t_2$). This follows from the $su(2)$ commutation relation $[t_1, t_2] = t_3$: since $[\mathrm{Ad}(c)t_1, \mathrm{Ad}(c)t_2] = \mathrm{Ad}(c)t_3$ then $C_H(\mathcal{N}) \subseteq C_H(u(1))$ and (41) is implied.

Now we note that

$$C_H(\mathcal{N}) = C_H(X), \quad X \in \mathcal{N}. \tag{42}$$

This follows from again using $C_H(\mathcal{N}) \subseteq C_H(u(1))$, which implies $[u(1), C_H(\mathcal{N})] = 0$. Then, because any $X' \in \mathcal{N}$ is proportional to $\mathrm{Ad}(h)X$ for some $h \in U(1)$, we infer that $C_H(X') = C_H(X)$ and obtain (42).

Therefore, by (41) and (42), each fully/asymptotically embedded monopole is an element in a set

$$\mathcal{O} \cong \frac{H}{C_H(X)}, \quad X \in \mathcal{N}. \tag{43}$$

For embedded monopoles (43) is exact and quantifies their H-degeneracy. For asymptotically embedded monopoles (43) refers only to their degeneracy at infinity and further degeneracy may occur in the core.

Finally, we note there is part of the orbit (43) that corresponds to spatial rotations. This is seen by considering the action of $h(\chi) = \exp(t_3\chi)$ on an $su(2)$ embedding

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos\chi & \sin\chi \\ -\sin\chi & \cos\chi \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}. \tag{44}$$

By (16) this is entirely equivalent to a spatial rotation. We comment that this relates to the angular momentum of the monopole [10,11].

5. Similarities with embedded vortices

Similar arguments have also been applied to embedded vortices [5]. These are $U(1) \to \mathbf{1}$ Nielsen–Olesen vortices [12] embedded according to (7)

$$\begin{array}{ccc} G & \to & H \\ \cup & & \cup \\ U(1) & \to & \mathbf{1} \end{array} \qquad \begin{aligned} &\Phi(r,\theta) = f(r)D\big(e^{\theta X}\big)\Phi_0, \\ &\boldsymbol{A}(r,\theta) = \frac{g(r)}{r}X\hat{\boldsymbol{\theta}}, \end{aligned} \tag{45}$$

with $U(1) = \exp(X\theta)$ and $X \in \mathcal{M}$. It was found that each vortex has its embedding constrained by [5]

$$X \in \mathcal{M}_a, \qquad D\big(e^{2\pi X}\big)\Phi_0 = 1, \tag{46}$$

with $\mathcal{M}_a$ one of the irreducible subspaces in Eq. (6). By similar arguments to those in Section 4 these have degenerate solution sets

$$\mathcal{O} = \mathrm{Ad}(H)X \cong \frac{H}{C_H(X)}. \tag{47}$$

For further discussion we refer to Ref. [5].

An interesting point is that the solution sets of embedded vortices and monopoles are very similar: with obvious parallels between Eqs. (31) and (46) and the orbits (43) and (47). Such parallels arise from the following sequence of embeddings

$$\begin{array}{ccccc} u(1) & \subseteq & su(2) & \subseteq & \mathcal{G} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \subseteq & u(1) & \subseteq & \mathcal{H} \end{array} \tag{48}$$

We also mention that similar considerations apply to whether vortices are fully embedded or asymptotically embedded—this will be discussed elsewhere [13].

6. Relation to fundamental monopoles

For the following discussion we restrict the scalar field to an adjoint representation of G. Then $\mathrm{rank}(G) =$

$\mathrm{rank}(H) = r$ and one may choose a maximal Abelian subgroup $T \subset H$ with orthonormal generators $\{T_1, \dots, T_r\}$.

Recall that the magnetic generator M of a non-Abelian monopole satisfies a topological constraint $\exp(4\pi M) = 1$. It can then be shown that these generators have a general form [2,9]

$$M = \sum_{a=1}^{r} n_a \boldsymbol{\beta}^*_{(a)} \cdot \boldsymbol{T}, \quad \boldsymbol{\beta}^*_{(a)} = \frac{\boldsymbol{\beta}_{(a)}}{\beta_a^2}, \tag{49}$$

where each n_a is an integer and $\{\boldsymbol{\beta}_{(1)}, \dots, \boldsymbol{\beta}_{(r)}\}$ are simple roots. These simple roots span the set of all roots $\boldsymbol{\alpha} \in \Phi(G)$

$$i\,\mathrm{ad}(\boldsymbol{T}) E_\alpha = \boldsymbol{\alpha} E_\alpha, \tag{50}$$

where each $\{E_\alpha\}$ is a different root space.

A fundamental monopole is a monopole that has its magnetic generator M associated with a simple root, so that $M = \boldsymbol{\beta}^*_{(a)} \cdot \boldsymbol{T}$ in (49). They are called fundamental because any non-Abelian monopole can be decomposed into several such monopoles [3]—this is supported both by (49) and index theory methods in the BPS limit.

In Ref. [3] fundamental monopoles were seen to take an asymptotic Prasad–Sommerfield form on the $su(2)_\alpha$ subtheory that has generators

$$\begin{aligned} t^1_\alpha &= (E_\alpha + E_{-\alpha})/\sqrt{2\alpha^2}, \\ t^2_\alpha &= -i(E_\alpha - E_{-\alpha})/\sqrt{2\alpha^2}, \\ t^3_\alpha &= \boldsymbol{\alpha}^* \cdot \boldsymbol{T} = \frac{\boldsymbol{\alpha}}{\alpha^2} \cdot \boldsymbol{T}. \end{aligned} \tag{51}$$

(Here each $E_{\pm\alpha}$ pair is normalized to $[E_\alpha, E_{-\alpha}] = i\boldsymbol{\alpha} \cdot \boldsymbol{T}$ so that $[t^a_\alpha, t^b_\alpha] = \epsilon_{abc} t^c_\alpha$.) Clearly, these are related to the $su(2)$ embeddings in (15), but how does the classification of these fundamental monopoles relate to the embedding condition $\mathcal{N} \subseteq \mathcal{M}_a$ in (31)?

To answer this question we define

$$\begin{aligned} su(2)_\alpha &= u(1)_\alpha \oplus \mathcal{N}_\alpha, \\ u(1)_\alpha &= \mathbb{R}\boldsymbol{\alpha} \cdot \boldsymbol{T}. \end{aligned} \tag{52}$$

Now a symmetry breaking $G \to T$ (for suitable Φ_0) has a decomposition like (6) into $\mathrm{Ad}(T)$-irreducible subspaces

$$\mathcal{G} = \mathcal{T} \oplus \sum_{\alpha \in \Phi(\mathcal{G})} \mathcal{N}_\alpha, \tag{53}$$

with $\mathcal{T}$ the Lie algebra of T. Then the action of $\mathrm{Ad}(T)$ upon each N_α gives simply an $SO(2)$ rotation R

$$\mathrm{Ad}\big(\exp(\boldsymbol{\theta} \cdot \boldsymbol{T})\big) \begin{pmatrix} t^1_\alpha \\ t^2_\alpha \end{pmatrix} = R(\boldsymbol{\alpha} \cdot \boldsymbol{\theta}) \begin{pmatrix} t^1_\alpha \\ t^2_\alpha \end{pmatrix}. \tag{54}$$

Since $T \subseteq H$ then also $\mathrm{Ad}(T) \subseteq \mathrm{Ad}(H)$, and each $\mathcal{M}_a$ splits into several $\mathcal{N}_\alpha$ components from (53). Therefore,

$$\begin{aligned} \mathcal{M}_a &= \sum_{\alpha \in \Phi(\mathcal{M}_a)} \mathcal{N}_\alpha, \\ \Phi(\mathcal{M}_a) &= \big\{\boldsymbol{\alpha} : \mathcal{N}_\alpha \subseteq \mathcal{M}_a\big\}. \end{aligned} \tag{55}$$

From (55) we see how the spectrum of fundamental monopoles relates to the constraint (31): $\mathcal{M}_a$ fragments into components N_α, $\boldsymbol{\alpha} \in \Phi(\mathcal{M}_a)$, of which each component gives a (possible) fundamental monopole embedding. Out of each set $\Phi(\mathcal{M}_a)$ an appropriate number of fundamental monopoles are taken.[4]

We now examine the monopoles that are embedded but not fundamental. For instance consider

$$\begin{array}{ccc} \mathcal{H} \oplus \mathcal{M} & \to & \mathcal{H} \\ \cup & & \cup \\ su(2)_{\alpha(1)} \times \cdots \times su(2)_{\alpha(p)} & \to & u(1)_{\alpha(1)} \times \cdots \times u(1)_{\alpha(p)} \end{array}$$

with $\{\boldsymbol{\alpha}_{(1)}, \dots, \boldsymbol{\alpha}_{(p)}\}$ mutually orthogonal roots. Then we interpret the diagonal $su(2)$ embedding

$$\begin{aligned} t^{1,2} &= t^{1,2}_{\alpha(b)} + t^{1,2}_{\alpha(c)}, \\ t^3 &= \big(\boldsymbol{\alpha}^*_{(b)} + \boldsymbol{\alpha}^*_{(c)}\big) \cdot \boldsymbol{T} \end{aligned} \tag{56}$$

as relating to a combination of fundamental monopoles. By (31), this embedding only gives a solution if both $\boldsymbol{\alpha}_{(b)}$ and $\boldsymbol{\alpha}_{(c)}$ are within the same $\Phi(\mathcal{M}_a)$.

7. Discussion

To conclude we summarize our results and make some comments:

(a) Our arguments relate to the following decomposition

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{M}, \quad \mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n,$$

with each $\mathcal{M}_a$ irreducible under $\mathrm{Ad}(H)$.

[4] We expect there to be $\mathrm{rank}(\mathcal{M}_a)$ fundamental monopoles in each $\mathcal{M}_a$, although proof is beyond the scope of this Letter.

(b) Monopole embeddings are constructed by

$$\begin{array}{ccccc} \mathcal{G} & = & \mathcal{H} & \oplus & \mathcal{M} \\ \cup & & \cup & & \cup \\ su(2) & = & u(1) & \oplus & \mathcal{N} \end{array}$$

and satisfy a constraint

$$\mathcal{N} \subset \mathcal{M}_a,$$

with $\mathcal{M}_a$ any subspace in (a).

Such monopoles can either be fully or asymptotically embedded (depending upon whether the embedded ansatz (16) solves the field equations everywhere or just asymptotically); this is determined by the representation of the scalar field and the scalar and gauge boson masses.

(c) The $su(2)$ embedding of each monopole lies in a set

$$\mathcal{O} \cong \mathrm{Ad}(H)su(2) \cong \frac{H}{C_H(X)}, \quad X \in \mathcal{N},$$

that is formed by acting the set of rigid H transformations on the monopole's long range magnetic field.

(d) Both the constraint in (b) and the degeneracy in (c) are analogous to those for embedded vortices. We interpret this as a consequence of the $su(2)$ monopole embedding containing some $u(1)$ vortex embeddings.

(e) The above relates to fundamental monopoles through

$$\mathcal{M}_a = \sum_{\alpha \in \Phi(\mathcal{M}_a)} N_\alpha,$$

$$su(2)_\alpha = u(1)_\alpha \oplus \mathcal{N}_\alpha,$$

where each $\mathcal{M}_a$ fragments into components $\mathcal{N}_\alpha$ that represent fundamental monopole embeddings.

(f) Finally, we expect the gauge degeneracy in (c) is related to the spectrum of zero modes, because the tangent space to $\mathcal{O}$ describes small, linear deformations that leave the monopole's energy unchanged. It is interesting that these deformations fall into multiplets of $C_H(X)$, which is itself closely related to the group of globally allowed gauge transformations $C_H(M)$.

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